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Ultradiscretization of solvable one-dimensional chaotic maps

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Abstract
We consider the ultradiscretization of a solvable one-dimensional chaotic map which arises from the duplication formula of the elliptic functions. It is shown that the ultradiscrete limit of the map and its solution yield the tent map and its solution simultaneously. A geometric interpretation of the dynamics of the tent map is given in terms of the tropical Jacobian of a certain tropical curve. Generalization to the maps corresponding to the mth multiplication formula of the elliptic functions is also discussed.

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1. Introduction

In this paper, we consider the following map,

$$z_{n+1} = f(z_n) = \frac{4z_n(1-z_n)(1-k^2z_n)}{\left(1-k^2z_n^2\right)^2}, \tag{1.1}$$

which admits the general solution

$$z_n = \mathrm{sn}^2(2^n u_0; k), \tag{1.2}$$

describing the orbit in [0, 1]. Here $\mathrm{sn}(u; k)$ is Jacobi's sn function, $0 < k < 1$ is the modulus and u_0 is an arbitrary constant. In fact, (1.1) can be reduced to the duplication formula of sn function:

$$\mathrm{sn}(2u; k) = \frac{2\mathrm{sn}(u; k)\mathrm{cn}(u; k)\,\mathrm{dn}(u; k)}{1-k^2\,\mathrm{sn}^4(u; k)}, \tag{1.3}$$

$$\mathrm{cn}^2(u; k) = 1 - \mathrm{sn}^2(u; k), \qquad \mathrm{dn}^2(u; k) = 1 - k^2\,\mathrm{sn}^2(u; k), \tag{1.4}$$

where $\mathrm{cn}(u;k)$ and $\mathrm{dn}(u;k)$ are Jacobi's cn and dn functions, respectively. The map (1.1) is a generalization of the logistic map (or the Ulam–von Neumann map):

$$z_{n+1} = 4z_n(1-z_n), \qquad z_n = \sin^2(2^n u_0). \tag{1.5}$$

Map (1.1) was first considered by Schröder [28] in 1871, and it has been studied by many authors [10, 12, 38, 39]. It is now classified as one of the (flexible) Lattès maps [21]. In this paper, we call (1.1) the *Schröder map*.

It is well known that the Schröder map is conjugate to the tent map for $X_n \in [0,1]$:

$$X_{n+1} = T_2(X_n) = 1 - 2\left|X_n - \frac{1}{2}\right| = \begin{cases} 2X_n & 0 \leqslant X_n \leqslant \frac{1}{2}, \\ 2(1-X_n) & \frac{1}{2} \leqslant X_n \leqslant 1. \end{cases} \tag{1.6}$$

Namely, we have the relation

$$s \circ f \circ s^{-1} = T_2, \qquad s(z) = \frac{1}{K(k)}\,\mathrm{sn}^{-1}(\sqrt{z};k), \tag{1.7}$$

where $K(k)$ is the complete elliptic integral of the first kind,

$$K(k) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{1.8}$$

The purpose of this paper is to establish a new relationship between the Schröder map and the tent map through a certain limiting procedure called the *ultradiscretization* [35]. The method of ultradiscretization has achieved a great success in the theory of integrable systems. From the integrable difference equations, various interesting piecewise linear dynamical systems have been constructed systematically, such as the soliton cellular automata [4, 6, 17, 22, 29–31, 34, 37, 41] and piecewise linear version of the Quispel–Roberts–Thompson (QRT) maps [23, 26, 32, 36]. The resulting piecewise linear discrete dynamical systems can be expressible in terms of the max and $\pm$ operations, which we call the ultradiscrete systems. The key of the method is that one can obtain not only the equations but also their solutions simultaneously. It also allows us to understand the underlying mathematical structures of the ultradiscrete systems [2, 3, 5, 8, 11, 13–16, 25, 33].

In this paper, we apply the ultradiscretization to the Schröder map (1.1) and its elliptic solution (1.2). As a result, they are reduced to the tent map and its solution. We also clarify the tropical geometric nature of the tent map; we show that the tent map can be regarded as the duplication map on the Jacobian of a certain tropical curve.

2. The ultradiscretization of the Schröder map

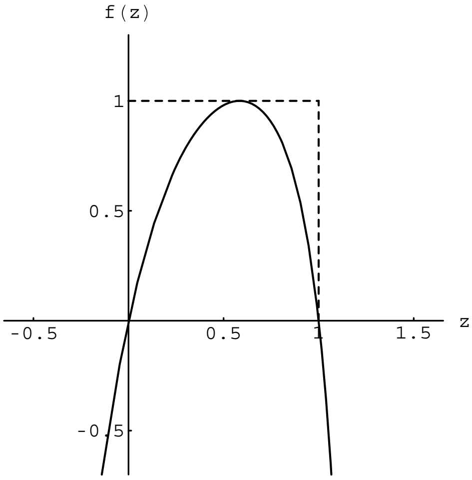
The key of the ultradiscretization is the following formula:

$$\lim_{\epsilon \to +0} \epsilon \log\left(\mathrm{e}^{\frac{A}{\epsilon}} + \mathrm{e}^{\frac{B}{\epsilon}} + \cdots\right) = \max(A, B, \ldots), \tag{2.1}$$

where the terms in log must be positive, and the dominant term survives under the limit. We note that the orbit of the map (1.1) is always restricted in [0, 1] if the initial value is in this interval. Since this is somewhat too restrictive for ultradiscretization, we apply the fractional linear transformation

$$z_n \longmapsto x_n = \frac{z_n}{1-z_n}, \tag{2.2}$$

which maps $[0,1] \to [0,\infty)$. Then the Schröder map (1.1) and its solution (1.2) are rewritten as



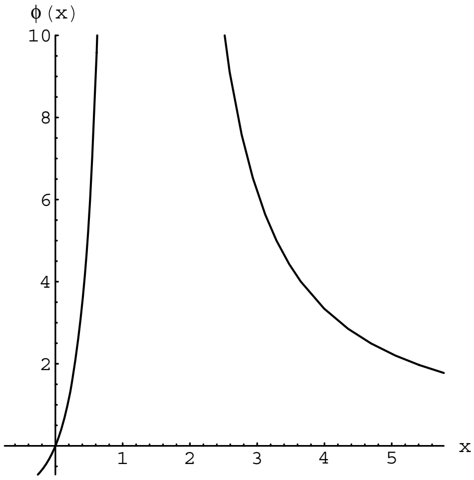


Figure 1. Map functions of (1.1) (left: $k = 0.7$) and (2.3) (right: $k' = 0.8$).

$$x_{n+1} = \phi(x_n) = \frac{4x_n(1+x_n)(1+k'^2x_n)}{\left(1-k'^2x_n^2\right)^2}, \qquad k'^2 = 1-k^2, \tag{2.3}$$

$$x_n = \frac{z_n}{1-z_n} = \frac{\mathrm{sn}^2(2^n u_0;k)}{1-\mathrm{sn}^2(2^n u_0;k)} = \frac{\mathrm{sn}^2(2^n u_0;k)}{\mathrm{cn}^2(2^n u_0;k)}, \tag{2.4}$$

respectively. We note that the map (2.3) can be obtained from (1.1) by replacing as $z_n \longrightarrow -x_n, k \longrightarrow k' = \sqrt{1-k^2}$. On the level of solution, this corresponds to Jacobi's imaginary transformation

$$-\mathrm{i}\,\mathrm{sn}(\mathrm{i}u;k') = \frac{\mathrm{sn}(u;k)}{\mathrm{cn}(u;k)}. \tag{2.5}$$

Figure 1 shows the map functions of (1.1) and (2.3). Note that $f(z)$ and $\phi(x)$ have poles at $z = \pm 1/k$ and $x = \pm 1/k'$, respectively.

Now we put

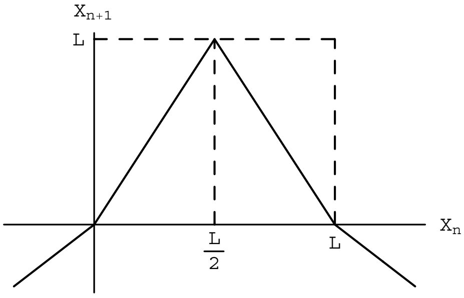
$$x_n = \exp\left[\frac{X_n}{\epsilon}\right], \qquad k' = \exp\left[-\frac{L}{2\epsilon}\right], \qquad (0 < k' < 1, L > 0). \tag{2.6}$$

Then (2.3) is rewritten as

$$X_{n+1} = F_\epsilon(X_n) = \epsilon \log\left[\frac{4\,\mathrm{e}^{\frac{X_n}{\epsilon}}\left(1+\mathrm{e}^{\frac{X_n}{\epsilon}}\right)\left(1+\mathrm{e}^{\frac{X_n-L}{\epsilon}}\right)}{\left(1-\mathrm{e}^{\frac{2X_n-L}{\epsilon}}\right)^2}\right]. \tag{2.7}$$

Taking the limit $\epsilon \to +0$ by using the formula (2.1), we obtain

$$\begin{aligned} X_{n+1} = F(X_n) &= X_n + \max(0, X_n) + \max(0, X_n - L) - 2\max(0, 2X_n - L) \\ &= \begin{cases} X_n & X_n < 0, \\ 2X_n & 0 \leqslant X_n < \frac{L}{2}, \\ -2X_n + 2L & \frac{L}{2} \leqslant X_n < L, \\ -X_n + L & L \leqslant X_n. \end{cases} \end{aligned} \tag{2.8}$$



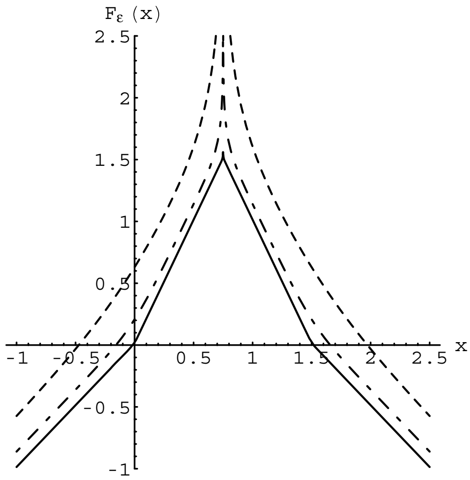


Figure 2. Left: map function of the ultradiscrete Schröder map (2.8). Right: limit transition of the map function $F_\epsilon(X)$ for $L = 1.5$. Dashed line: $\epsilon = 0.3$, dot-dashed line: $\epsilon = 0.1$, solid line: $\epsilon = 0.01$.

Remark 2.1. Although the terms in log in the formula (2.1) must be positive in general, the negative terms can also exist as long as they are not dominant in the limit. For example, we have

$$\lim_{\epsilon\to+0} \epsilon \log\left(e^{\frac{A}{\epsilon}} - e^{\frac{B}{\epsilon}}\right)^2 = \lim_{\epsilon\to+0} \epsilon \log\left(e^{\frac{2A}{\epsilon}} - 2\,e^{\frac{A+B}{\epsilon}} + e^{\frac{2B}{\epsilon}}\right) = 2\max(A, B). \tag{2.9}$$

We call the map (2.8) the *ultradiscrete Schröder map*. Figure 2 shows the map function of (2.8) and limit transition of the function $F_\epsilon(X)$. The dynamics of the map (2.8) is described as follows: if the initial value X_0 is in $[0, L]$, the map is the tent map and $X_n \in [0, L]$ for all n. If $X_0 \in (-\infty, 0]$, then $X_n = X_0$ for all $n \geqslant 1$. Finally if $X_0 \in [L, \infty)$, then $X_1 = -X_0 + L < 0$ and $X_n = X_1$ for all $n \geqslant 1$. Therefore, the ultradiscrete Schröder map (2.8) is essentially the tent map on $[0, L]$:

$$X_{n+1} = L\left(1 - 2\left|\frac{X_n}{L} - \frac{1}{2}\right|\right), \qquad X_n \in [0, L], \tag{2.10}$$

and otherwise the dynamics is trivial.

Now let us consider the limit of the solution by using the ultradiscretization of the elliptic theta functions [32](see also [14, 24, 25]). Jacobi's elliptic functions are expressed in terms of the elliptic theta functions $\vartheta_i(v)$ $(i = 0, 1, 2, 3)$ as

$$\mathrm{sn}(u; k) = \frac{\vartheta_3(0)\vartheta_1(v)}{\vartheta_2(0)\vartheta_0(v)}, \qquad \mathrm{cn}(u; k) = \frac{\vartheta_0(0)\vartheta_2(v)}{\vartheta_2(0)\vartheta_0(v)}, \tag{2.11}$$

$$u = \pi(\vartheta_3(0))^2 v, \qquad k^2 = \left(\frac{\vartheta_2(0)}{\vartheta_3(0)}\right)^4, \tag{2.12}$$

where

$$\vartheta_0(v) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} z^{2n}, \tag{2.13}$$

$$\vartheta_1(v) = \mathrm{i}\sum_{n\in\mathbb{Z}} (-1)^n q^{(n-1/2)^2} z^{2n-1}, \tag{2.14}$$

$$\vartheta_2(\nu) = \sum_{n\in\mathbb{Z}} q^{(n-1/2)^2} z^{2n-1}, \tag{2.15}$$

$$\vartheta_3(\nu) = \sum_{n\in\mathbb{Z}} q^{n^2} z^{2n}, \tag{2.16}$$

and $z = \exp[\mathrm{i}\pi\nu]$. We parametrize the nome q as

$$q = \exp\left[-\frac{\epsilon\pi^2}{\theta}\right], \qquad \theta > 0. \tag{2.17}$$

Applying Jacobi's imaginary transformation (or Poisson's summation formula) the elliptic theta functions are rewritten as

$$\vartheta_0(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n\in\mathbb{Z}} \exp\left[-\frac{\theta}{\epsilon}\left\{\nu - \left(n + \frac{1}{2}\right)\right\}^2\right], \tag{2.18}$$

$$\vartheta_1(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n\in\mathbb{Z}} (-1)^n \exp\left[-\frac{\theta}{\epsilon}\left\{\nu - \left(n + \frac{1}{2}\right)\right\}^2\right], \tag{2.19}$$

$$\vartheta_2(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n\in\mathbb{Z}} (-1)^n \exp\left[-\frac{\theta}{\epsilon}(\nu - n)^2\right], \tag{2.20}$$

$$\vartheta_3(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n\in\mathbb{Z}} \exp\left[-\frac{\theta}{\epsilon}(\nu - n)^2\right]. \tag{2.21}$$

Asymptotic behaviour of these functions for $\epsilon \to +0$ is given by

$$\vartheta_0(0) \sim 2\sqrt{\frac{\theta}{\epsilon\pi}} \exp\left[-\frac{\theta}{4\epsilon}\right], \tag{2.22}$$

$$\vartheta_2(0) \sim \sqrt{\frac{\theta}{\epsilon\pi}} \left(1 - 2\exp\left[-\frac{\theta}{\epsilon}\right]\right), \tag{2.23}$$

$$\vartheta_3(0) \sim \sqrt{\frac{\theta}{\epsilon\pi}} \left(1 + 2\exp\left[-\frac{\theta}{\epsilon}\right]\right), \tag{2.24}$$

$$(\vartheta_0(\nu))^2 \sim \frac{\theta}{\epsilon\pi} \exp\left[-\frac{2\theta}{\epsilon}\left\{((\nu)) - \frac{1}{2}\right\}^2\right], \tag{2.25}$$

$$(\vartheta_1(\nu))^2 \sim \frac{\theta}{\epsilon\pi} \exp\left[-\frac{2\theta}{\epsilon}\left\{((\nu)) - \frac{1}{2}\right\}^2\right], \tag{2.26}$$

$$(\vartheta_2(\nu))^2 \sim \frac{\theta}{\epsilon\pi} \left(\exp\left[-\frac{\theta}{\epsilon}\{((\nu))\}^2\right] - \exp\left[-\frac{\theta}{\epsilon}\{((\nu)) - 1\}^2\right]\right)^2, \tag{2.27}$$

where $((\nu))$ is the decimal part of ν, namely,

$$((\nu)) = \nu - \mathrm{Floor}(\nu), \qquad 0 \leqslant ((\nu)) < 1. \tag{2.28}$$

Then we have

$$k'^2 = \exp\left[-\frac{L}{\epsilon}\right] = 1 - k^2 = 1 - \left(\frac{\vartheta_2(0)}{\vartheta_3(0)}\right)^4 \sim \frac{16\exp\left[-\frac{\theta}{\epsilon}\right]\left(1 + 4\exp\left[-\frac{2\theta}{\epsilon}\right]\right)}{\left(1 + 2\exp\left[-\frac{\theta}{\epsilon}\right]\right)^4},$$

$$x_n = \exp\left[\frac{X_n}{\epsilon}\right] = \frac{\mathrm{sn}^2(u;k)}{\mathrm{cn}^2(u;k)} = \left(\frac{\vartheta_3(0)\vartheta_1(\nu)}{\vartheta_0(0)\vartheta_2(\nu)}\right)^2 \sim \frac{\left(1 + 2\exp\left[-\frac{\theta}{\epsilon}\right]\right)^2 \exp\left[\frac{2\theta((\nu))}{\epsilon}\right]}{4\left(1 - \exp\left[\frac{\theta}{\epsilon}[2((\nu)) - 1]\right]\right)^2},$$

which yield in the limit $\epsilon \to +0$

$$L = \theta, \tag{2.29}$$

$$X_n = \theta\left(1 - 2\left|((\nu)) - \tfrac{1}{2}\right|\right), \qquad \nu = 2^n \nu_0, \tag{2.30}$$

respectively, where ν_0 is an arbitrary constant. We note that in taking the limit of x_n, we have put the arbitrary constant u_0 as

$$u_0 = \frac{\theta}{\epsilon}\nu_0 \tag{2.31}$$

so that

$$\nu = \frac{2^n u_0}{\pi(\vartheta_3(0))^2} = 2^n \nu_0 \frac{\frac{\theta}{\epsilon}}{\pi(\vartheta_3(0))^2} \longrightarrow 2^n \nu_0 \qquad (\epsilon \to +0). \tag{2.32}$$

One can verify that (2.29) and (2.30) actually satisfy the ultradiscrete Schröder map (2.8) or (2.10) by direct calculation. Therefore, we have shown that through the ultradiscretization the Schröder map (2.3) and its solution (2.4) yield the map (2.8) (or (2.10)) and its solution (2.30) simultaneously.

Remark 2.2

(i) The fundamental periods of $\frac{\mathrm{sn}^2(u;k)}{\mathrm{cn}^2(u;k)}$ are $2K(k)$ and $2\mathrm{i}K(k')$. In the ultradiscretization of the elliptic theta functions, we have parametrized the nome q as (2.17), which implies that the ratio of half-period τ is given by $\tau = \mathrm{i}\frac{\epsilon\pi}{\theta}$ and

$$K(k) = \frac{\pi}{2}(\vartheta_3(0))^2 \sim \frac{\theta}{2\epsilon}, \qquad K(k') = -\frac{\pi \mathrm{i}}{2}(\vartheta_3(0))^2\tau = \frac{\pi^2\epsilon}{2\theta}(\vartheta_3(0))^2 \sim \frac{\pi}{2}, \tag{2.33}$$

as $\epsilon \to +0$. Since we have $u = \frac{\theta}{\epsilon}\nu$, the fundamental periods with respect to ν tend to 1 and $\frac{\mathrm{i}\epsilon\pi}{\theta}$ as $\epsilon \to +0$. This implies that the ultradiscretization of the elliptic functions is realized by collapsing the imaginary period and keeping the real period finite.

(ii) The Schröder map (1.1) is reduced to the logistic map (1.5) for $k = 0$. This corresponds to the ultradiscrete Schröder map (2.8) with $L = 0$,

$$X_{n+1} = -|X_n|, \tag{2.34}$$

whose dynamics is trivial, and the solution (2.30) becomes $X_n = 0$. Therefore ultradiscretization of the logistic map does not yield an interesting map [9]. In fact, we see that this case is not consistent with the ultradiscrete limit, since the asymptotic behaviour of $K(k)$ and $K(k')$ as $k \to 0$ is given by

$$K(k) \sim \frac{\pi}{2}, \qquad K(k') \sim \log\frac{4}{k}. \tag{2.35}$$

One can apply the same procedure to the following map which originates from the triplication formula of sn^2[12, 21, 39]:

$$z_{n+1} = g(z_n) = \frac{z_n\left\{k^4 z_n^4 - 6k^2 z_n^2 + 4(k^2+1)z_n - 3\right\}^2}{\left\{3k^4 z_n^4 - 4k^2(k^2+1)z_n^3 + 6k^2 z_n^2 - 1\right\}^2}, \qquad z_n = \mathrm{sn}^2(3^n u_0; k), \tag{2.36}$$

which is rewritten as

$$x_{n+1} = \gamma(x_n) = \frac{x_n\left\{k'^4 x_n^4 - 6k'^2 x_n^2 - 4(k'^2+1)x_n - 3\right\}^2}{\left\{3k'^4 x_n^4 + 4k'^2(k'^2+1)x_n^3 + 6k'^2 x_n^2 - 1\right\}^2}, \qquad x_n = \frac{\mathrm{sn}^2(3^n u_0; k)}{\mathrm{cn}^2(3^n u_0; k)}, \tag{2.37}$$

by the transformation (2.2). The map functions $g(z)$ and $\gamma(x)$ are illustrated in figure 3.

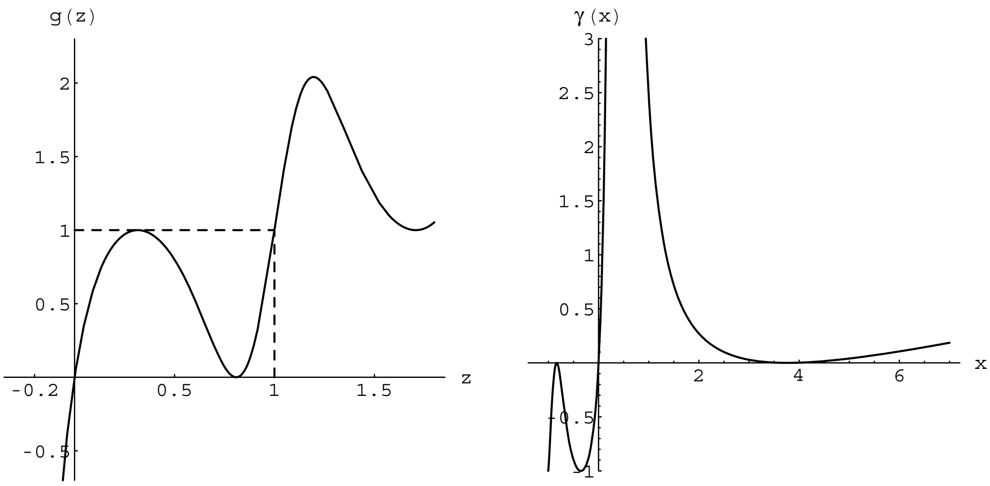


Figure 3. Map functions of (2.36) (left: $k = 0.7$) and (2.37) (right: $k' = 0.8$)

Then the ultradiscretization of (2.37) yields the map

$$
\begin{aligned}
X_{n+1} = G(X_n) &= X_n + 2\max(0, X_n, 4X_n - 2L) - 2\max(0, 3X_n - L, 4X_n - 2L) \\
&= \begin{cases} X_n & X_n < 0, \\ 3X_n & 0 \leqslant X_n < \frac{L}{3}, \\ -3X_n + 2L & \frac{L}{3} \leqslant X_n < \frac{2L}{3}, \\ 3X_n - 2L & \frac{2L}{3} \leqslant X_n < L, \\ X_n & L \leqslant X_n, \end{cases}
\end{aligned}
\tag{2.38}
$$

and its solution

$$X_n = L\left(1 - 2\left|((\nu)) - \tfrac{1}{2}\right|\right), \qquad \nu = 3^n \nu_0. \tag{2.39}$$

Figure 4 shows the map function $G(X_n)$ and the limit transition of the map function of

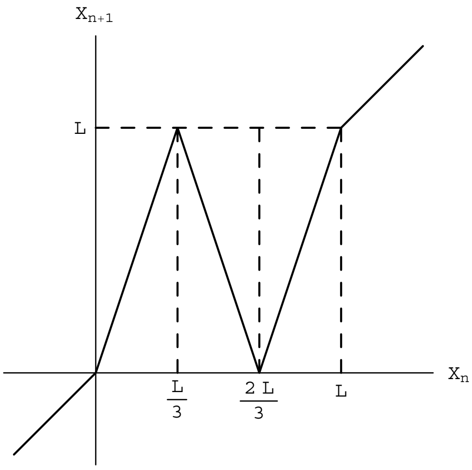
$$X_{n+1} = G_\epsilon(X_n) = \epsilon \log\left[\frac{\mathrm{e}^{\frac{X_n}{\epsilon}}\left\{\mathrm{e}^{\frac{4X_n-2L}{\epsilon}} - 6\,\mathrm{e}^{\frac{2X_n-L}{\epsilon}} - 4\left(\mathrm{e}^{-\frac{L}{\epsilon}} + 1\right)\mathrm{e}^{\frac{X_n}{\epsilon}} - 3\right\}^2}{\left\{3\,\mathrm{e}^{\frac{4X_n-2L}{\epsilon}} + 4\left(\mathrm{e}^{-\frac{2L}{\epsilon}} + \mathrm{e}^{-\frac{L}{\epsilon}}\right)\mathrm{e}^{\frac{3X_n}{\epsilon}} + 6\,\mathrm{e}^{\frac{2X_n-L}{\epsilon}} - 1\right\}^2}\right]. \tag{2.40}$$

We note that one can directly ultradiscretize the map (2.36) to obtain (2.38); however, the solution $x_n = \mathrm{sn}^2(3^n u_0; k)$ degenerates to the trivial solution $X_n = 0$. Thus it is important to consider (2.37) in order to obtain the limit which is consistent with the solution.

It is possible to apply ultradiscretization to the maps arising from the mth multiplication formula of sn^2 [12, 21] in a similar manner.

3. Geometric description in terms of the tropical geometry

It is shown in [5, 25] that the tropical geometry provides a geometric framework for the description of the ultradiscrete integrable systems. Therefore, it may be natural to expect that a similar framework also works well for our case. In this section, we show that the ultradiscrete Schröder map can be interpreted as the duplication map on the Jacobian of a certain tropical curve. As for the basic notions of the tropical geometry, we refer to [1, 7, 27].



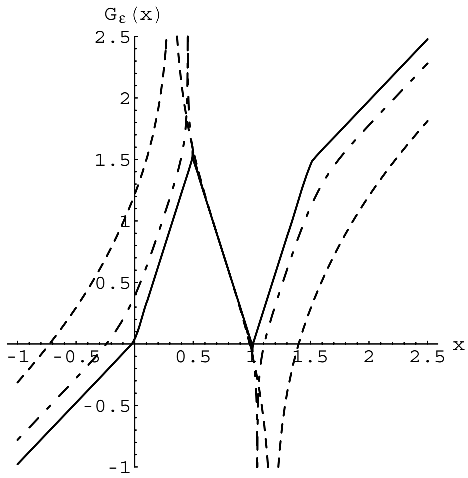


Figure 4. Left: map function of the map (2.38). Right: limit transition of the map function $G_\epsilon(X)$ for $L = 1.5$. Dashed line: $\epsilon = 0.3$, dot-dashed line: $\epsilon = 0.1$, solid line: $\epsilon = 0.01$.

We first consider the elliptic curve

$$[xy - b(x + y) + c]^2 = 4d^2xy, \tag{3.1}$$

parametrized by

$$(x, y) = \left(\frac{\mathrm{sn}^2(u; k)}{\mathrm{cn}^2(u; k)}, \frac{\mathrm{sn}^2(u + \eta; k)}{\mathrm{cn}^2(u + \eta; k)}\right), \tag{3.2}$$

where η is a constant and b, c, d are given by

$$b = \frac{1}{k'^2}\frac{\mathrm{cn}^2(\eta; k)}{\mathrm{sn}^2(\eta; k)}, \qquad c = \frac{1}{k'^2}, \qquad d = -\frac{1}{k'^2}\frac{\mathrm{dn}(\eta; k)}{\mathrm{sn}^2(\eta; k)}, \tag{3.3}$$

respectively. Eliminating η in (3.3), we see that b and d satisfy the relation

$$k'^2d^2 = (1 + k'^2b)(1 + b). \tag{3.4}$$

We may regard the Schröder map (2.3) as the projection of the dynamics of the point on the elliptic curve (3.1) to the x-axis.

We next apply the ultradiscretization to the elliptic curve. Putting

$$\begin{aligned} &x = \mathrm{e}^{\frac{X}{\epsilon}}, \qquad y = \mathrm{e}^{\frac{Y}{\epsilon}}, \qquad b = \mathrm{e}^{\frac{B}{2\epsilon}}, \qquad 4d^2 = \mathrm{e}^{\frac{D}{\epsilon}}, \\ &k' = \mathrm{e}^{-\frac{L}{2\epsilon}}, \qquad c = \frac{1}{k'^2} = \mathrm{e}^{\frac{L}{\epsilon}}, \qquad L > 0, \end{aligned} \tag{3.5}$$

and taking the limit $\epsilon \to +0$, (3.1) and (3.4) yield

$$\max(2X + 2Y, B + 2X, B + 2Y, 2L) = X + Y + D, \tag{3.6}$$

and

$$-L + D = \max\left(0, \frac{B}{2} - L\right) + \max\left(0, \frac{B}{2}\right), \tag{3.7}$$

respectively. The condition (3.7) gives the following three cases:

$$\text{(i)} \quad B > 2L > 0, \quad D = B, \tag{3.8}$$

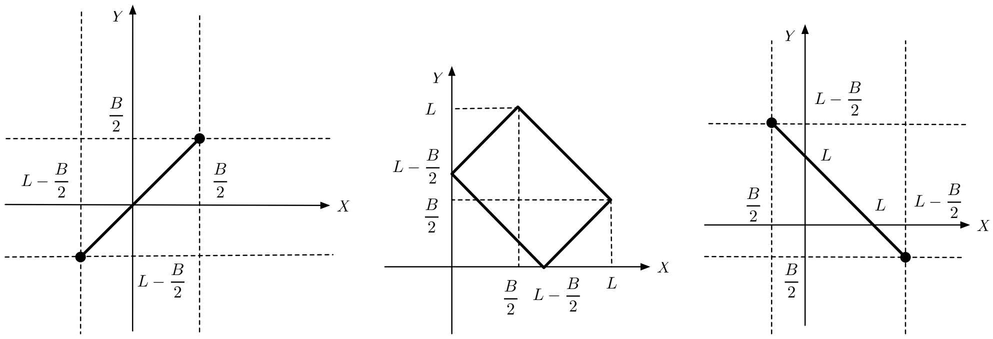


Figure 5. Ultradiscretization of the elliptic curve (3.1). Left: case (i), centre: case (ii), right: case (iii).

$$\text{(ii)}\quad 2L > B > 0, \qquad D = L + \frac{B}{2}, \tag{3.9}$$

$$\text{(iii)}\quad 0 > B, \qquad D = L. \tag{3.10}$$

For each case, the set of points defined by (3.6) is (i) a line connecting $\left(\frac{B}{2}, \frac{B}{2}\right)$ and $\left(L - \frac{B}{2}, L - \frac{B}{2}\right)$, (ii) a rectangle with vertices $\left(0, L - \frac{B}{2}\right)$, $\left(L - \frac{B}{2}, 0\right)$, $\left(L, \frac{B}{2}\right)$ and $\left(\frac{B}{2}, L\right)$, (iii) a line connecting $\left(\frac{B}{2}, L - \frac{B}{2}\right)$ and $\left(L - \frac{B}{2}, \frac{B}{2}\right)$, respectively, as illustrated in figure 5. In the following, we consider only the case (ii) and denote the rectangle as $\overline{C}$.

Let us recall some notions of the tropical geometry. The tropical curve defined by the tropical polynomial

$$\Xi(X, Y) = \max_{(a_1,a_2)\in\mathcal{A}} (\lambda_{(a_1,a_2)} + a_1X + a_2Y), \qquad \mathcal{A} \in \mathbb{Z}^2, \tag{3.11}$$

is a set of points $(X, Y) \in \mathbb{R}^2$ where Ξ is not smooth. Here $\mathcal{A}$ is a finite subset of $\mathbb{Z}^2$ called the support, and we denote as $\Delta(\mathcal{A})$ the convex hull of $\mathcal{A}$. Let Γ_d be the triangle in $\mathbb{Z}^2$ with vertices $(0, 0)$, $(d, 0)$, $(0, d)$. Then the degree of the tropical curve is d if $\Delta(\mathcal{A})$ is inside Γ_d but not inside Γ_{d-1} [40]. The genus of the tropical curve is defined as the first Betti number of the curve, namely the number of its cycles [1, 18, 19].

We consider the tropical polynomial

$$\Psi(X, Y) = \max(2X + 2Y, B + 2X, B + 2Y, 2L, X + Y + D), \tag{3.12}$$

under the condition (3.9). Let C be the tropical curve defined by Ψ, which is illustrated in figure 6. Then the degree and the genus of C are 4 and 1, respectively. Note that the rectangle $\overline{C}$ is exactly the cycle of C.

Vigeland [40] has successfully introduced the group law on the tropical elliptic curve. Unfortunately, however, his definition of tropical elliptic curve is limited to 'smooth' curve of degree 3 and hence it does not cover our case. Nevertheless, it is possible to define the tropical Jacobian $J(\overline{C})$ of C [5, 20] and characterize the dynamics of the ultradiscrete Schröder map (2.10) on it in the following manner: let V_i and E_i $(i = 1, \ldots, 4)$ be the vertices and edges of $\overline{C}$ defined by

$$\begin{aligned} V_1 &= \mathcal{O} = \left(0, L - \frac{B}{2}\right), & V_2 &= \left(L - \frac{B}{2}, 0\right), \\ V_3 &= \left(L, \frac{B}{2}\right), & V_4 &= \left(\frac{B}{2}, L\right), \end{aligned} \tag{3.13}$$

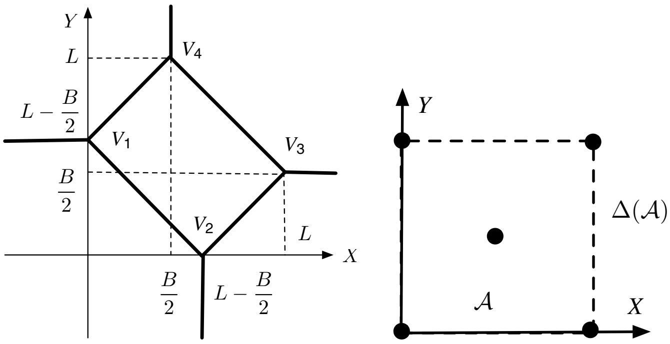


Figure 6. Left: tropical curve C defined by (3.12). Right: support of (3.12).

$$V_1V_2 = E_1, \qquad V_2V_3 = E_2, \qquad V_3V_4 = E_3, \qquad V_4V_1 = E_4, \tag{3.14}$$

respectively. The length of each edge is given as

$$|E_1| = \sqrt{2}\left(L - \frac{B}{2}\right), \qquad |E_2| = \frac{\sqrt{2}}{2}B, \qquad |E_3| = \sqrt{2}\left(L - \frac{B}{2}\right), \qquad |E_4| = \frac{\sqrt{2}}{2}B. \tag{3.15}$$

The primitive tangent vector for each edge is

$$\boldsymbol{v}_1 = (1, -1), \qquad \boldsymbol{v}_2 = (1, 1), \qquad \boldsymbol{v}_3 = (-1, 1), \qquad \boldsymbol{v}_4 = (-1, -1). \tag{3.16}$$

We introduce the total lattice length $\mathcal{L}$ as the sum of the length of each edge scaled by the length of the corresponding primitive tangent vector, which is computed as

$$\mathcal{L} = \sum_{i=1}^{4} \frac{|E_i|}{|\boldsymbol{v}_i|} = 2L. \tag{3.17}$$

Then the tropical Jacobian $J(\overline{C})$ is defined by

$$J(\overline{C}) = \mathbb{R}/\mathcal{L}\mathbb{Z} = \mathbb{R}/2L\mathbb{Z}. \tag{3.18}$$

The Abel–Jacobi map $\mu : \overline{C} \to J(\overline{C})$ is defined as the piecewise linear map which is linear on each edge satisfying

$$\mu(V_1) = 0, \qquad \mu(V_2) = L - \frac{B}{2}, \qquad \mu(V_3) = L, \qquad \mu(V_4) = 2L - \frac{B}{2}. \tag{3.19}$$

Let $\pi : \overline{C} \to \mathbb{R}$ be the projection of the point on $\overline{C}$ to the X-axis. Let ρ be the map defined by $\rho = \pi \circ \mu^{-1} : J(\overline{C}) \to \mathbb{R}$ which maps $\mu(P)$ $(P \in \overline{C})$ to the X-coordinate of P. Here we note that π^{-1} is 1:2 and we define $\pi^{-1}(X)$ to be the point on $\overline{C}$ whose Y-coordinate is smaller. In this setting, $\rho(p)$ $(p \in J(\overline{C}))$ can be written as

$$\rho(p) = (\pi \circ \mu^{-1})(p) = \begin{cases} p & 0 \leqslant p \leqslant L, \\ -p + 2L & L \leqslant p \leqslant 2L, \end{cases} \tag{3.20}$$

as shown in the left of figure 7.

Now we define the duplication map $\varphi_2 : J(\overline{C}) \to J(\overline{C})$ by

$$\varphi_2(p) \equiv 2p (\mathrm{mod}\, \mathcal{L}), \qquad p \in J(\overline{C}), \tag{3.21}$$

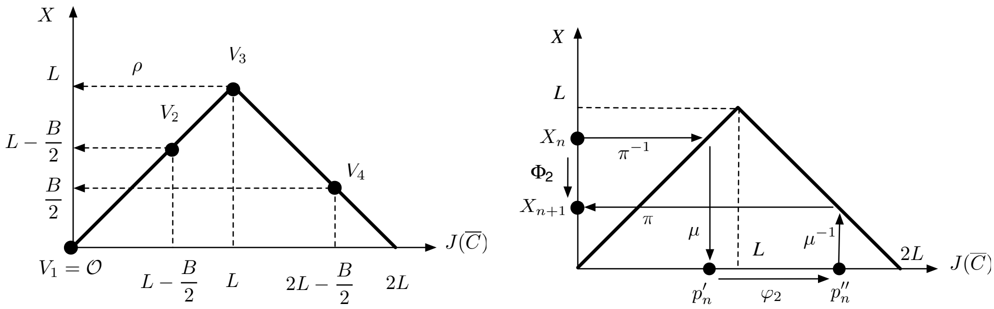


Figure 7. Left: correspondence between X and $J(\overline{C})$ by ρ. Right: duplication map φ_2 and Φ_2.

and introduce $\Phi_2 : \mathbb{R} \to \mathbb{R}$ as the conjugation map of φ_2 by ρ,

$$\Phi_2 = \rho \circ \varphi_2 \circ \rho^{-1}. \tag{3.22}$$

In order to write the map Φ_2 explicitly, we introduce $p', p'' \in J(\overline{C})$ for $P = (X, Y) \in \overline{C}$ by

$$p' = \rho^{-1}(X) = (\mu \circ \pi^{-1})(X) = X, \qquad p'' = \varphi_2(p') = 2p' = 2X. \tag{3.23}$$

Then the map Φ_2 is expressed as follows (the right of figure 7):

(i) For $0 \leqslant X \leqslant \frac{L}{2}$: since $0 \leqslant p'' \leqslant L$, (3.20) implies

$$\Phi_2(X) = \rho(p'') = 2X. \tag{3.24}$$

(ii) For $\frac{L}{2} \leqslant X \leqslant L$: since $L \leqslant p'' \leqslant 2L$, (3.20) implies

$$\Phi_2(X) = \rho(p'') = -2X + 2L. \tag{3.25}$$

The dynamical system

$$X_{n+1} = \Phi_2(X_n) = L\left(1 - 2\left|\frac{X_n}{L} - \frac{1}{2}\right|\right) = \begin{cases} 2X_n & 0 \leqslant X \leqslant \frac{L}{2}, \\ -2X_n + 2L & \frac{L}{2} \leqslant X \leqslant L, \end{cases} \tag{3.26}$$

coincides with the ultradiscrete Schröder map (2.10). Therefore, we have shown that the ultradiscrete Schröder map (2.10) can be regarded as the duplication map on the Jacobian $J(\overline{C})$ of the tropical curve C defined by the tropical polynomial (3.12).

Similarly, we define the triplication map $\varphi_3 : J(\overline{C}) \to J(\overline{C})$ by

$$\varphi_3(p) \equiv 3p (\mathrm{mod}\, \mathcal{L}), \qquad p \in J(\overline{C}), \tag{3.27}$$

and introduce $\Phi_3 : \mathbb{R} \to \mathbb{R}$ as the conjugation map of φ_3 by ρ,

$$\Phi_3 = \rho \circ \varphi_3 \circ \rho^{-1}. \tag{3.28}$$

Then the corresponding dynamical system is given by

$$X_{n+1} = \Phi_3(X_n) = \begin{cases} 3X_n & 0 \leqslant X_n \leqslant \frac{L}{3}, \\ -3X_n + 2L & \frac{L}{3} \leqslant X \leqslant \frac{2L}{3}, \\ 3X_n - 2L & \frac{2L}{3} \leqslant X \leqslant L \end{cases}$$
$$= 3X_n - 2\max(0, 3X_n - L) + 2\max(0, 3X_n - 2L), \tag{3.29}$$

which is equivalent to (2.38) on $[0, L]$. For general m, the mth multiplication map yields the dynamical system

$$X_{n+1} = \Phi_m(X_n) = mX_n + 2\sum_{i=1}^{m-1}(-1)^i \max(0, mX_n - iL), \tag{3.30}$$

which may be regarded as the ultradiscretization of the map arising from the mth multiplication formula of $\frac{\mathrm{sn}^2}{\mathrm{cn}^2}$.

4. Concluding remarks

In this paper, we have presented a new relationship between two typical chaotic one-dimensional maps, the Schröder map and the tent map, through the ultradiscretization. Although the ultradiscretization has been developed in the theory of integrable systems, the results in this paper imply the possibility of applying the method to a wider class of dynamical systems. Our results also suggest that the tropical geometry combined with the ultradiscretization provides a powerful tool to study a piecewise linear map, since the ultradiscretization translates the geometric background of the original rational map into that of the corresponding piecewise linear map. It would be an interesting problem to study various ultradiscrete or piecewise linear systems, such as ultradiscrete analogues of Painlevé systems, generalized QRT maps and higher-dimensional solvable chaotic maps in this direction.

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